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Order Compactifications of Totally Ordered Topological Spaces*

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In [3], as an application of two basic theorems on inclusions and lattice cones of functions, we gave an intrinsic description of the order compactifications of an ordered topological space in terms of certain quasiproximities. Of the questions "When does an ordered topological space have a largest, a smallest, and a (up to equivalence) unique order compactification?" we answered only the first. These results were announced in [2]. They are reproduced here in the first section in a form free of the language of quasiproximities used in [2] and [3]. In the second section we study the order compactifications of a totally ordered topological space and answer the remaining two of the above questions for this special case.

1. ORDERED TOPOLOGICAL SPACES

1.1. DEFINITION. An *ordered topological space* is a triple $(X, \tau, \leqslant)$ consisting of a set X, a topology τ for X, and an order $\leqslant$ for X, i.e., a transitive, reflexive, antisymmetric binary relation $\leqslant$ for X, which is *closed*, i.e., $\leqslant$ is a closed subset of $X \times X$ when the latter is given the product topology.

1.2. PROPOSITION. The topology of an ordered topological space is Hausdorff.

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Proof. [6, p. 27].

1.3. DEFINITION. Let X be an ordered topological space (whenever possible, we shall omit explicit mention of the topology of an ordered topological space and shall denote the order by $\leqslant$). An *order compactification* of X is a pair (Y, κ) consisting of a *compact ordered topological space* Y, i.e., an ordered topological space Y whose topology is compact and a mapping $\kappa: X \to Y$ such that

κ is a topological embedding,

κ is an *order embedding*, i.e., $\kappa(x) \leqslant \kappa(y)$ iff $x \leqslant y$,

$\kappa[X]$ is dense in Y.

An order compactification (Y, κ) of X is called a *Nachbin compactification* if its order is the smallest closed order for Y with respect to which κ is an order embedding.

We define a *preorder*, i.e., a transitive and reflexive binary relation, $\leqslant$ for the set of all order compactifications of X by agreeing that $(Y_1, \kappa_1) \leqslant (Y_2, \kappa_2)$ iff there exists a mapping $\varphi: Y_2 \to Y_1$ such that

φ is continuous,

φ is increasing, i.e., $x \leqslant y \Rightarrow \varphi(x) \leqslant \varphi(y)$,

$\varphi \circ \kappa_2 = \kappa_1$,

and we call two order compactifications (Y_1, κ_1) and (Y_2, κ_2) of X *equivalent* if there exists a mapping $\varphi: Y_2 \to Y_1$ such that

φ is a topological isomorphism,

φ is an order isomorphism,

$\varphi \circ \kappa_2 = \kappa_1$.

It follows from 1.2 that two order compactifications (Y_1, κ_1) and (Y_2, κ_2) of X are equivalent iff $(Y_1, \kappa_1) \leqslant (Y_2, \kappa_2)$ and $(Y_2, \kappa_2) \leqslant (Y_1, \kappa_1)$.

For any set X, we denote by $B(X)$ the space of all bounded real-valued functions on X equipped with the topology of uniform convergence and we call a *closed lattice cone* (*of functions*) *on* X any closed convex cone in $B(X)$ which contains the constant functions and is closed under the lattice operations. If X is a topological space, $BC(X)$ denotes the closed lattice cone on X which consists of all continuous members of $B(X)$ and if X is an ordered topological space, $BCI(X)$ denotes the closed lattice cone on X which consists of all continuous and increasing members of $B(X)$.

1.4. DEFINITION. Let X be an ordered topological space. We shall call a closed lattice cone K on X *compatible* if the topology of X is the smallest topology for X which renders the members of K continuous and the order of X is the largest order for X which renders the members of K increasing; it is easy to see that K is compatible iff

$K \subset BCI(X)$,

for every $x \in X$ and every neighborhood U of x there exists $f \in K$ and $g \in -K$ such that $1_{\{x\}} \leqslant f \wedge g \leqslant 1_U$ (= characteristic function of U),

$x \leqslant y$ if (and only if) $f(x) \leqslant f(y)$ for all $f \in K$.

We call X a *completely regular ordered topological space* if $BCI(X)$ is compatible.

1.5. ORDER COMPACTIFICATION THEOREM. *Let X be an ordered topological space.*

(i) *X has an order compactification iff it is a completely regular ordered topological space.*

(ii) *If (Y, κ) is an order compactification of X, then $\{f \circ \kappa : f \in BCI(Y)\}$ is a compatible closed lattice cone on X and is called the closed lattice cone associated with (Y, κ).*

(iii) *If K is a compatible closed lattice cone on X, the set Y of all maps $y: K \to \mathbf{R}$ such that*

y is additive and $\mathbf{R}^+$-homogeneous,

y preserves the lattice operations,

y preserves the constants, i.e., for $r \in \mathbf{R}$, $y(r1_X) = r$, equipped with the topology of pointwise convergence and with the pointwise order together with the map $\kappa: X \to Y$ defined by

$$\kappa(x)(f) = f(x), \qquad f \in K, \qquad x \in X,$$

is a compactification of X whose associated closed lattice cone is K.

(iv) *If (Y_1, κ_1) and (Y_2, κ_2) are two order compactifications of X with associated closed lattice cones K_1 and K_2, then*

$$(Y_1, \kappa_1) \leqslant (Y_2, \kappa_2) \qquad \textit{iff} \qquad K_1 \subset K_2 .$$

1.6. NACHBIN COMPACTIFICATION THEOREM. *Let X be an ordered topological space, (Y, κ) an order compactification of X, and let K be the associated closed lattice cone.*

(i) *The binary relation* $\leqslant'$ *for* Y *defined by* $x \leqslant' y \Leftrightarrow f(x) \leqslant f(y)$ *for all* $f \in BC(Y)$ *such that* $f \circ \kappa \in BCI(X)$ *is the smallest closed order for* Y *with respect to which* κ *is an order embedding.*

(ii) $\{f \circ \kappa : f \in BC(Y)\} = cl(K - K)$.

(iii) (Y, κ) *is a Nachbin compactification iff* $K = cl(K - K) \cap BCI(X)$.

1.7. THEOREM. *Let* X *be a completely regular ordered topological space. Then every nonempty set of order compactifications resp. Nachbin compactifications of* X *has a smallest upper bound (unique up to equivalence) in the preordered set of all order compactifications resp. Nachbin compactifications. In particular,* X *has a largest order compactification (unique up to equivalence) and this order compactification is a Nachbin compactification.*

The part of this theorem which concerns Nachbin compactifications was actually not proved in [3]. It is, however, straightforward to verify that for any nonempty set of Nachbin compactifications of X a smallest upper bound in the preordered set of all Nachbin compactifications of X can be constructed from a smallest upper bound in the preordered set of all order compactifications by changing the order of the latter according to 1.6(i).

2. TOTALLY ORDERED TOPOLOGICAL SPACES

Let X be a set and let $\leqslant$ be a *total order* for X, i.e., an order for X such that for $x, y \in X$, $x \leqslant y$ or $y \leqslant x$. As usual, we define the ($\leqslant$ —) *order topology* of X to be the smallest topology for X which contains all sets of the form $\{y : y < x\ (y \leqslant x \text{ and } y \neq x)\}$ and $\{y : x < y\}$ for some $x \in X$. It is an easy exercise to show that this is the smallest topology for X with respect to which the order is closed in the sense of 1.1. This fact can be stated as follows.

2.1. PROPOSITION. *The topology of a totally ordered topological space, i.e., an ordered topological space whose order is total, is larger than the order topology.*

The next proposition is an immediate consequence of 2.1, 1.2 and the definitions involved.

2.2. PROPOSITION. *Let* X *be a totally ordered topological space and let* (Y, κ) *be an order compactification of* X. *Then the order of* Y *is total, the topology of* Y *is the order topology and* (Y, κ) *is a Nachbin compactification.*

This proposition should be contrasted with the following consequence of 1.6:

Let X be an ordered topological space whose order is equality and let (Y, κ) be an order compactification of X. Then (Y, κ) is a Nachbin compactification iff the order of Y is equality.

Let X be an ordered topological space. A subset A of X is said to be *increasing* (resp. *decreasing*) if $x \in A$ and $x \leqslant y$ (resp. $y \leqslant x$) imply $y \in A$. Clearly, the set of increasing subsets of X and the set of decreasing subsets of X are closed under arbitrary unions and intersections. For $A \subset X$ we shall denote by $cl_i(A)$ (resp. $cl_d(A)$) the intersection of all closed increasing (resp. decreasing) subsets of X which contain A. For $x \in X$, $cl_i(\{x\}) = \{y: x \leqslant y\}$ and $cl_d(\{x\}) = \{y: y \leqslant x\}$. X is called a *normally ordered topological space* if whenever F_1 and F_2 are disjoint closed subsets of X such that F_1 is increasing and F_2 is decreasing, there are disjoint open sets U_1 and U_2 such that U_1 is increasing and contains F_1 and U_2 is decreasing and contains F_2. A subset A of X is said to be *convex* if $x, y \in A$ and $x \leqslant z \leqslant y$ imply $z \in A$. X is said to be a *locally convex ordered topological space* if for every $x \in X$ and every neighborhood U of X there exists a convex neighborhood of x which is contained in U. It follows from 1.4 that X is a locally convex ordered topological space if X is a completely regular ordered topological space. The converse is obviously false in general (take, for example, the order of X to be equality). We shall see, however, that it is true if the order of X is total.

2.3. LEMMA. *A totally ordered topological space is a normally ordered topological space.*

Proof. Let F_1 and F_2 be disjoint closed subsets of X such that F_1 is increasing and F_2 is decreasing. If $F_1 \cup F_2 = X$, then both F_1 and F_2 are open. If there exists an $x \in X \sim (F_1 \cup F_2)$, then $F_1 \subset \{y: x < y\}$ and $F_2 \subset \{y: y < x\}$. By 2.1 we are done.

2.4. PROPOSITION. *Let X be a totally ordered topological space. Then X is a completely regular ordered topological space iff it is a locally convex ordered topological space.*

Proof. only if. See the remark preceding 2.3.

if. Let $x \in X$ and let U be an open neighborhood of x. Let V be an open convex neighborhood of x which is contained in U and set $A = (X \sim V) \cap cl_i(\{x\})$ and $B = (X \sim V) \cap cl_d(\{x\})$. A is a closed increasing subset of X disjoint from $cl_d(\{x\})$. By 2.3 and [6, p. 30] there exists $g \in BCI(X) \ni 1_A \leqslant g \leqslant 1_{X \sim cl_d(\{x\})}$. Likewise, B is a closed decreasing subset of X disjoint from $cl_i(\{x\})$ and so there exists $f \in BCI(X) \ni 1_{cl_i(\{x\})} \leqslant f \leqslant 1_{X \sim B}$. It follows that $1_{\{x\}} \leqslant f \wedge (1 - g) \leqslant 1_{X \sim (A \cup B)} \leqslant 1_V$.

Now let x and y be in X such that $y < x$. Then $cl_i(\{x\})$ and $cl_d(\{y\})$ are disjoint and as before there exists $f \in BCI(X) \ni 1_{cl_i(\{x\})} \leqslant f \leqslant 1_{X \sim cl_d(\{y\})}$. In particular, $f(y) = 0 < 1 = f(x)$.

2.5. LEMMA. *Let X be a totally ordered topological space. Then for any nonempty subset C of X the maps φ_C, ψ_C: $BCI(X) \to \mathbf{R}$ defined by*

$$\left.\begin{aligned} \varphi_C(f) &= \inf f[C] \\ \psi_C(f) &= \sup f[C] \end{aligned}\right., \qquad f \in BCI(X),$$

are additive and $\mathbf{R}^+$-homogeneous, preserve the lattice operations, and preserve the constants.

Proof. That φ_C and ψ_C are $\mathbf{R}^+$-homogeneous and preserve the constants, is clear; that they are additive and preserve the lattice operations, follows from the fact that for any f and g in $BCI(X)$ and any two points x and y of C, say $x \leqslant y$, $f(x) \square g(x) \leqslant f(x) \square g(y) \leqslant f(y) \square g(y)$ where $\square$ is any of $+$, $\vee$, $\wedge$.

2.6. DEFINITION. Let X be a totally ordered topological space. We shall denote by $\Gamma(X)$ the set of all subsets C of X which have the properties

$\varnothing \subsetneqq C \subsetneqq X$,

C is closed, open and increasing,

C has no infimum in X.

We note that the last property is equivalent to

C has no smallest element and $X \sim C$ has no largest element.

2.7. THEOREM. *Let X be a completely regular totally ordered topological space, let K be a compatible closed lattice cone on X and let γ be a subset of $\Gamma(X)$.*

(i) $K = \{f \in BCI(X)\colon \inf f[C] \leqslant \sup f[X \sim C]$ *for all* $C \in \Gamma(X) \sim \gamma_K\}$ *where* $\gamma_K = \{C \in \Gamma(X)\colon 1_C \in K\}$.[1]

(ii) $K_\gamma = \{f \in BCI(X)\colon \inf f[C] \leqslant \sup f[X \sim C]$ *for all* $C \in \Gamma(X) \sim \gamma\}$ *is a compatible closed lattice cone on X and* $\gamma = \{C \in \Gamma(X)\colon 1_C \in K_\gamma\}$.

Proof.

(i) If $C \in \Gamma(X) \sim \gamma_K$, then $1_C \notin K$ and so (see Footnote 1)

[1] Note the trivial fact that for any lattice cone K on any set X, given two nonempty subsets A and B of X, $\inf f[A] > \sup f[B]$ for some $f \in K$ iff there exists $f \in K$ such that $1_A \leqslant f \leqslant 1_{X \sim B}$; in particular, if $B = X \sim A$, $\inf f[A] > \sup f[X \sim A]$ for some $f \in K$ iff $1_A \in K$.

$\inf f[C] \leqslant \sup f[X \sim C]$ for all $f \in K$. Thus $K \subset \{f \in BCI(X): \inf f[C] \leqslant \sup f[X \sim C]$ for all $C \in \Gamma(X) \sim \gamma_K\}$.

To prove the converse inclusion, let $f \in BCI(X)$ be such that $\inf f[C] \leqslant \sup f[X \sim C]$ for all $C \in \Gamma(X) \sim \gamma_K$. By [3, 4.1(i)] f belongs to K if for any two nonempty subsets A and B of X such that $\inf f[A] > \sup f[B]$, there exists a $g \in K$ such that $\inf g[A] > \sup g[B]$. So, let A and B be two such sets. Clearly $cl_i(A) \cap cl_d(B) = \varnothing$. If

$$cl_i(A) = X \sim cl_d(B) \quad \text{and} \quad cl_i(A) \in \Gamma(X),$$

then $1_{cl_i(A)} \in K$. If

$$\text{not}(cl_i(A) = X \sim cl_d(B) \text{ and } cl_i(A) \in \Gamma(X)),$$

then either

$$\exists x \in X \sim cl_d(B) \ni cl_i(A) \subset cl_i(\{x\}), \tag{1}$$

or

$$\exists x \in X \sim cl_i(A) \ni cl_d(B) \subset cl_d(\{x\}) \tag{2}$$

(if for every $x \in X \sim cl_d(B)$, there exists $y \in cl_i(A)$ such that $y < x$, then $cl_i(A) = X \sim cl_d(B)$ and $cl_i(A)$ has no smallest element, and so $cl_d(B)$ has a largest element x). If (1) holds, there exist (observe that K is compatible) $g_1 \in K$ and $g_2 \in -K$ such that $1_{\{x\}} \leqslant g_1 \wedge g_2 \leqslant 1_{X \sim cl_d(B)}$. Since $x \leqslant y$ for all $y \in cl_i(A)$, g_1 is 1 on $cl_i(A)$. Since $y < x$ for all $y \in cl_d(B)$, g_2 is 1 on $cl_d(B)$ and so g_1 is 0 on $cl_d(B)$. Thus $\inf g_1[A] = 1 > 0 = \sup g_1[B]$. The argument in the case that (2) holds is analogous. This completes the proof of (i).

(ii) By 2.5 K_γ is a closed lattice cone on X and we show next that $\gamma = \{C \in \Gamma(X): 1_C \in K_\gamma\}$. Let $C \in \Gamma(X)$. Clearly, $C \in \gamma$ if $1_C \in K_\gamma$. If, conversely, $C \in \gamma$, then $1_C \in K_\gamma$: $1_C \in BCI(X)$ and for any $C' \in \Gamma(X) \sim \gamma$, $\inf 1_C[C'] = 1 = \sup 1_C[X \sim C']$ if $C \sim C'$ is nonempty and $\inf 1_C[C'] = 0 = \sup 1_C[X \sim C']$ if $C' \sim C$ is nonempty. Thus it remains to show that K_γ is compatible. To do so, it is obviously sufficient to show that the closed lattice cone

$$K_\varnothing = \{f \in BCI(X): \inf f[C] \leqslant \sup f[X \sim C] \text{ for all } C \in \Gamma(X)\},$$

is compatible and this, in turn, will be shown if we can exhibit an order compactification (Y, κ) of X with the property that $K_\varnothing$ contains the closed lattice cone associated with (Y, κ).[2] We proceed to construct (Y, κ). Set

[2] We shall see later (2.8) that this order compactification is a smallest order compactification of X.

$\mathscr{C} = \{C \subset X$: C is closed, open and increasing and either C has no smallest element or $X \sim C$ has no largest element$\}$ (observe that $\mathscr{C}$ contains both $\varnothing$ and X) and let Y_0 be the disjoint union of X and $\mathscr{C}$. It is straightforward to verify that the definition

$$\begin{array}{llll} C \leqslant C' & \text{iff} \quad C \subset C', & C, C' \in \mathscr{C}, & \\ x \leqslant C & \text{iff} \quad x \in C, & x \in X, & C \in \mathscr{C}, \\ C \leqslant x & \text{iff} \quad x \in X \sim C, & x \in X, & C \in \mathscr{C}, \end{array}$$

extends the total order of X to a total order of Y_0. That the order topology of Y_0 is compact, is a mildly intricate exercise if one uses the well-known fact that the order topology of any totally ordered set is compact if (and only if) every subset has a supremum. Finally, it is easy to see that the inclusion of X in Y_0 is a topological (and order) embedding. Thus, the closure Y of X in Y_0 (Y does not contain $\varnothing$ or X according as X has a smallest element or a largest element) together with the inclusion map is an order compactification of X and, obviously, the closed lattice cone associated with this order compactification is contained in $K_\varnothing$.

Let X be a completely regular totally ordered topological space. By 2.7 the map $\gamma \mapsto K_\gamma$ is a preorder isomorphism of the power set of $\Gamma(X)$ onto the set of compatible lattice cones on X, both preordered by inclusion. By 1.5 the latter set is (modulo equivalence) isomorphic with the preordered set of all order compactifications of X. Thus the following two corollaries are obvious.

2.8. COROLLARY. *A completely regular totally ordered topological space has a smallest order compactification* (*e.g., the one constructed in the proof of* 2.7(ii)).

This corollary should be contrasted with the following well-known result of P. Alexandroff [1] and S. Fomin [4]:

Let X be an ordered topological space whose order is equality. Then X has a smallest Nachbin compactification iff its topology is locally compact and a Nachbin compactification (Y, κ) of X is a smallest Nachbin compactification iff $Y \sim \kappa[X]$ has at most one point.

2.9. COROLLARY. *Let X be a completely regular totally ordered topological space. Then X has a* (*up to equivalence*) *unique order compactification iff $\Gamma(X) = \varnothing$, i.e., every nonempty and proper subset of X which is closed, open and increasing, has an infimum in X.*

This corollary should be contrasted with the following result of E. Hewitt [5]: Let X be a completely regular ordered topological space whose order is equality. Then X has a (up to equivalence) unique Nachbin compactification iff of any two completely separated closed subsets of X one is compact.

2.10. PROPOSITION. *Let X be a completely regular totally ordered space and let γ be a subset of $\Gamma(X)$. Let K be the compatible closed lattice cone on X corresponding to γ by* 2.7(ii) *and let (Y, κ) be the order compactification of X corresponding to K by* 1.5(iii). *Then for $y \in Y \sim \kappa[X]$, $C = \{x \in X: y(f) \leq f(x)$ for all $f \in K\}$ is a closed, open and increasing subset of X which has at least one of the properties*

$$C \text{ is nonempty and has no smallest element and} \\ y(f) = \inf f[C] \quad \text{for all} \quad f \in K, \tag{1}$$

and

$$X \sim C \text{ is nonempty and has no largest element and} \\ y(f) = \sup f[C \sim X] \quad \text{for all} \quad f \in K, \tag{2}$$

and has both iff it belongs to $\Gamma(X) \sim \gamma$.

Proof. C is clearly closed and increasing. Since $C = \{x \in X: y(f) < f(x)$ for all $f \in K\}$, C is also open. Now let $\{x_i\}_{i \in I}$ be a net in X such that the net $\{\kappa(x_i)\}_{i \in I}$ converges to y. One of $I_C = \{i \in I: x_i \in C\}$ and $I_{X \sim C} = \{i \in I: x_i \in X \sim C\}$ is cofinal in I. If I_C is cofinal in I, then C is nonempty and has no smallest element. That $\{\kappa(x_i)\}_{i \in I_C}$ converges to y means just that $\{f(x_i)\}_{i \in I_C}$ converges to $y(f)$ for all $f \in K$ and this implies that $y(f) = \inf f[C]$ for all $f \in K$. Thus C has the property (1). Likewise, if $I_{X \sim C}$ is cofinal in I, C has the property (2).

If C has both properties, then $C \in \Gamma(X)$ and $C \notin \gamma$ by 2.7(ii). Conversely, if $C \in \Gamma(X) \sim \gamma$, then, by the definition of K, $\inf f[C] \leq \sup f[X \sim C]$ for all $f \in K$ and so (observe that always $\inf f[C] \geq y(f) \geq \sup f[X \sim C]$ for all $f \in K$) $\inf f[C] = y(f) = \sup f[X \sim C]$ for all $f \in K$. The proof is complete.

Observing 2.5, it is an immediate consequence of 2.10 that, under the assumptions of 2.10, the convex subsets of Y which are contained in $Y \sim \kappa[X]$ and have more than one element are precisely the two-point sets $\{\varphi_C, \psi_{X \sim C}\}$ for $C \in \gamma$ where φ_C and $\psi_{X \sim C}$ are as in 2.5. Thus we have another corollary.

2.11. COROLLARY. *Let X be a completely regular totally ordered topological space. Then X has a (up to equivalence) unique order compactification iff for a largest order compactification (Y, κ) of X, every convex subset of Y which is contained in $Y \sim \kappa[X]$ has at most one point.*

This corollary should be contrasted with another result of E. Hewitt [5]: Let X be a completely regular ordered topological space whose order is equality. Then X has a (up to equivalence) unique Nachbin compactification iff for a largest Nachbin compactification (Y, κ) of X, $Y \sim \kappa[X]$ has at most one point.

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